

## CHERN CLASSES AND PROJECTIVE GEOMETRY

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### 1. Introduction

Classical projective geometry is rich in relations between the extrinsic invariants (e.g., order, double tangents, number of nodes, triple points, ...) associated with an algebraic map  $f: M \rightarrow CP_N$  of a projective algebraic manifold  $M$ . It has also long been known that these extrinsic invariants may be sometimes used to define *birational* or *intrinsic* invariants of the manifold.

For example, the Plücker formulas for an algebraic plane curve may be interpreted as a definition of the 1st Chern class of an algebraic manifold of dimension 1, or the postulation formula as the arithmetic genus in terms of the projective characters of an algebraic surface.

The object of this note is:

(a) to obtain descriptions of the Chern classes of an algebraic manifold in terms of the extrinsic invariants associated with an algebraic map  $f: M \rightarrow CP_N$  which is nonsingular of order 1, i.e., the derivative of  $f$  has maximal rank everywhere,

(b) to show how the Chern classes affect well-known geometric invariants, associated with an imbedding satisfying the above condition.

### 2. The bundle of tangent spaces of a variety

Let  $f: M \rightarrow CP_N$  be an algebraic map which is nonsingular of order 1, and let  $\dim M = n$ .

**Definition.** The tangent projective space to  $f$  at  $x \in M$  is the unique linear space  $P$ , of dimension  $n$ , of  $CP_N$ , which passes through  $f(x)$  such that

$$\text{Im}(Df(x) \cdot (T_x M)) = T_{f(x)} P$$

where  $T_x M$  is the tangent space of  $M$  at  $x$ , and  $T_{f(x)} P$  is the tangent space of  $P$  at  $f(x)$ .

Given an algebraic map  $f: M \rightarrow CP_N$ , as above, we shall call  $f$  a *cuspid-free algebraic variety*. The set of tangent spaces of cuspid-free algebraic variety forms a fibre bundle (with fibre  $CP_n$ ) over  $M$ , and, in fact, may be realised as an

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algebraic variety by the construction sketched below. (For details see Pohl [2], [3].)

Let  $G_{n,N}$  be the Grassmann manifold of linear  $n$ -spaces of  $CP_N$ , and  $E_{n,N} \xrightarrow{\pi} G_{n,N}$  the tautologous bundle over  $G_{n,N}$ , with fibre  $CP_n$ . Let  $\tau: E_{n,N} \rightarrow CP_N$  be the map which takes a point  $x$  in an  $n$ -linear space  $P$  of  $CP_N$  to  $x$ .

Now there is a map  $f': M \rightarrow G_{n,N}$  called the *dual map*, associated with a cusp-free variety which associates to each  $x \in M$ , the tangent projective space at  $x$ .

Then the pull-back:

$$\begin{array}{ccc} P_f = (f')^*E_{n,N} & \xrightarrow{f'} & E_{n,N} \\ \pi_f \downarrow & & \downarrow \pi \\ M & \xrightarrow{f'} & G_{n,N} \end{array}$$

yields a  $CP_n$ -bundle  $P_f$ , over  $M$ , which is obviously in (1-1) correspondence with the set of points of the set of tangent projective spaces of  $M$ .

$P_f$  is called the *bundle of tangent-spaces* of  $f$ , and the map  $\tau \circ f': P_f \rightarrow CP_N$  realises this manifold as an algebraic variety.  $P_f$  is a  $CP_n$ -bundle associated to a holomorphic vector bundle  $E_f \rightarrow M$  of fibre dimension  $n + 1$ . Moreover,  $E_f$  is topologically, though not analytically, isomorphic to the bundle  $f^*\hat{l}_N \oplus (TM \otimes f^*\hat{l}_N)$ , where  $\hat{l}_N \rightarrow CP_N$  is the tautologous line bundle. Also,  $(\tau \circ f')^*\hat{l}_N$  is the tautologous line bundle  $\hat{l}_f$  over  $P_f$ . Let  $l_N \rightarrow CP_N$  and  $l_f \rightarrow P_f$  be the line bundles conjugate to  $\hat{l}_N \rightarrow CP_N$  and  $\hat{l}_f \rightarrow P_f$  respectively. Then by the Leray-Hirsch theorem, the map<sup>1</sup>

$$\varphi: H^*(M) \otimes H^*(CP_n) \rightarrow H^*(P_f)$$

defined by

$$\varphi(\alpha \otimes x^m) = \pi^*(\alpha) \cup \chi_f^m$$

is an isomorphism of  $H^*(M)$ -modules, where  $x \in H^2(CP_n)$  is the Poincaré dual of a hyperplane, and  $\chi_f$  is the 1st Chern class of  $l_f$ :  $\chi_f \in H^2(P_f)$ .

Moreover, in  $H^{2n+2}(P_f)$  we have

$$\chi_f^{n+1} = -\sum_{j=1}^{n+1} c_j(E_f) \cdot \chi_f^{n+1-j},$$

where  $c_j$  is the  $j$ -th Chern class.

<sup>1</sup> Here, as in what follows, we use rational coefficients for all cohomology groups.

### 3. Chern classes of varieties

Let  $E \rightarrow X$  be a complex vector bundle of dimension  $r$ , and let us factorise formally the polynomial

$$1 + c_1t + \dots + c_r t^r = (1 + \gamma_1 t) \cdots (1 + \gamma_r t),$$

where  $c_n = c_n(E)$  is the  $n$ -th Chern class of  $E$ . Then the polynomial

$$\chi_q(\gamma) = \sum_{\substack{\alpha_1 + \dots + \alpha_r = q \\ \alpha_i \geq 0}} \gamma^{\alpha_1} \cdots \gamma^{\alpha_r}$$

is clearly symmetric, and can thus be expressed in terms of the Chern classes. We denote this polynomial by  $\mathfrak{P}_q(E)$ . For example,  $\mathfrak{P}_1(E) = c_1$ ,  $\mathfrak{P}_2(E) = c_1^2 - c_2$ ,  $\mathfrak{P}_3(E) = c_1^3 - 2c_1c_2 + c_3$ .

**Theorem 1.** *Let  $f: M \rightarrow CP_N$  be a cusp-free algebraic variety,  $N \geq 2n$ ,  $\Delta_j$  be the set of points of  $M$  such that the tangent projective  $n$ -space at  $x \in \Delta_j$  meets a generically situated  $CP_{N-j} \subset CP_N$  ( $\Delta_j$  is a subvariety of  $M$ ), and  $\delta_j$  be the Poincaré dual of  $\Delta_j$ . Then for  $n \leq j \leq 2n$ ,*

$$\delta_j = \mathfrak{P}_{j-n}(E_f).$$

*Proof.*  $\Delta_j = \pi_j(\tau \cdot \bar{f})^{-1}(P_{N-j})$ ,  $P_{N-j}$  being generic. Now the genericity assumption implies that  $\tau \cdot \bar{f}: P_f \rightarrow CP_N$  is transversal to  $P_{N-j}$ . Hence the fundamental class of  $(\tau \cdot \bar{f})^{-1}(P_{N-j})$  is given by

$$\mathfrak{D}_{P_f}(\tau \cdot \bar{f})^*(x^j),$$

where  $\mathfrak{D}_{P_f}: H^*(P_f) \rightarrow H_*(P_f)$  is the Poincaré duality isomorphism. Hence

$$\delta_j = (\mathfrak{D}_M)^{-1} \cdot (\pi_f)_* \cdot (\mathfrak{D}_{P_f}) \cdot (\tau \cdot \bar{f})^*(x^j) = (\pi_f)_! (\tau \cdot \bar{f})^*(x^j) = (\pi_f)_!(\chi_f^j),$$

where  $(\pi_f)_!: H^*(P_f, M) \rightarrow H^*(M)$  is the “*umkehrungshomomorphismus*” or more simply “integration over the fibre”. Hence  $\delta_j$  is the coefficient of  $\chi_f^j$  in  $\chi_f^j$  written as a polynomial in  $c_j(E_f)$  and  $\chi_f^m$ , where the degree of the  $\chi_f$ -terms are  $\leq n$ .

The result follows from an easy calculation. q.e.d.

The Chern classes of  $E_f$  may be computed by well-known formulas (see Hirzebruch [1, p. 64] for example) in terms of the Chern classes of  $M$ , and  $\xi = f^*(x)$  where  $\xi$  is the class dual to a generic hyperplane section of  $f$ . We may then use these to compute the  $\delta_j$ 's successively from  $j = n + 1, \dots, 2n$ . These expressions for  $\delta_j$  can now be inverted to obtain expressions for  $c_j(M)$ , the Chern classes of  $M$ , in terms of  $\xi$  and  $\delta_j$ . The actual computations are complicated; so we give some examples:

**Theorem 2A.** *Let  $f: M \rightarrow CP_N, N \geq 4$ , be a cusp-free algebraic surface (i.e.,  $\dim_C M = 2$ ). Then*

$$c_1(M) = 3\xi - \delta_3, \quad c_2(M) = 3\xi^2 - 2\xi \cdot \delta_3 + \delta_3^2 - \delta_4,$$

where  $\xi, \delta_i$  are as above.

**Theorem 2B.** Let  $f: M \rightarrow CP_N$  ( $N \geq 6$ ) be a cusp-free variety  $\dim M = 3$ . Then

$$\begin{aligned} c_1(M) &= 4\xi - \delta_4, \\ c_2(M) &= 6\xi^2 - 3\xi \cdot \delta_2 + (\delta_4^2 - \delta_5), \\ c_3(M) &= 4\xi^3 - 3\xi^2 \cdot \delta_4 + 2\xi(\delta_4^2 - \delta_5) - 2\delta_4 \cdot \delta_5 + \delta_4^3 + \delta_6. \end{aligned}$$

**Remarks.** Our method is of course not the first attempt at a geometric formulation of the Chern classes. Indeed Chern classes were first introduced, in the Chow ring of a nonsingular variety by Todd and Eger using the so-called "canonical systems".

Our result is certainly more "geometric"-the classes  $\delta_j$  are far more intuitive than Todd's canonical classes, which are pullbacks of Schubert cycles under the dual map. Also our method yields results in arbitrary codimensions, provided the singularities of the variety are generic, i.e., those which would arise when a nonsingular variety of dimension  $n$  in  $CP_{2n+k}$  is projected to a  $CP_m$  ( $n \leq m$ ). In such a case the classes  $\delta_j$  may be reinterpreted in terms of other geometric invariants. For example, suppose a surface  $S$  in  $CP_3$  arises from a generic projection of  $F: M \rightarrow CP_5$  onto  $CP_3$ ,  $F$  being a cusp-free variety. Then  $\delta_4$  clearly does not make sense for  $S$ . However,  $\delta_4(F)$  is the Poincaré dual of the set of "pinch-points" in the double-curve of  $S$ . (See, Semple and Roth [4, p. 202]).

#### 4. Chern classes and extrinsic invariants

In this section we show how the Chern classes of an algebraic manifold  $M$  affect the extrinsic invariants of a cusp-free variety  $f: M \rightarrow CP_N$ . We discuss only algebraic surfaces, where the computations are still reasonable and yet there are a great wealth of results.

Let  $f: M \rightarrow CP_n$  be an algebraic surface;  $n \geq 5$ .

**Definition.** The projective characters of  $f$  are defined as follows:

The order  $\mu_0$  is the number of points at which a generic  $CP_{n-2}$  meets  $f$ . If  $\Gamma$  is the curve cut out on  $M$  by a generic hyperplane section, the rank  $\mu_1$  is the number of tangents of  $\Gamma$  which meet a generic  $CP_{n-2}$ . The class  $\mu_2$  is the number of hyperplanes belonging to a generic pencil which are tangent to  $f$ . The ceto or type  $\nu_2$  is the number of tangent planes which meet a generic  $CP_{n-4}$ .

**Theorem 3.** Let  $f: M \rightarrow CP_n$ ,  $n \geq 5$ , be a cusp-free surface,  $\xi \in H^2(M)$  be the class dual to a hyperplane section, and  $c_1, c_2$  be the 1st and 2nd Chern classes of  $M$ . Then

$$\begin{aligned} \mu_0 &= \langle \xi^2, [M] \rangle, \\ \mu_1 &= \langle 3\xi^2 - \xi \cdot c_1, [M] \rangle, \\ \mu_2 &= \langle 3\xi^2 - 2\xi \cdot c_1 + c_2, [M] \rangle, \\ \nu_2 &= \langle 6\xi^2 - 4\xi \cdot c_1 + c_1^2 - c_2, [M] \rangle, \end{aligned}$$

where  $[M]$  is the fundamental class, and  $\langle , \rangle$  the Kronecker pairing.

*Proof.* The formula for  $\mu_0$  is trivial, and that of  $\nu_2$  obvious, since  $\nu_2 = \langle \delta_4, [M] \rangle$ . To obtain the formulas for  $\mu_1, \mu_2$  we use the following result, which is very easy to prove (see Semple and Roth [4, p. 194] for example):

Let  $P, P'$  be two 3-codimensional linear spaces generically situated with respect to  $f$ , and  $A_3, A'_3$  be the algebraic varieties (defined in §3) which arise from  $P$  and  $P'$ . Then these are algebraic curves,  $\mu_1$  is the number of points of intersection of  $A_3$  with a generic hyperplane, and  $(\mu_1 + \nu_2)$  is the number of points of intersection of  $A_3$  and  $A'_3$ . Thus

$$\begin{aligned} \mu_1 &= \langle x, f_*(\mathbb{D}_M \cdot \delta_3) \rangle = \langle \xi, \mathbb{D}_M \cdot \delta_3 \rangle = \langle \xi, \delta_3 \cap [M] \rangle \\ &= \langle \xi \cup (3\xi - c_1), [M] \rangle = \langle 3\xi^2 - \xi \cdot c_1, [M] \rangle, \\ \mu_2 &= \langle \delta_3 \cup \delta'_3, [M] \rangle - \nu_2 \\ &= \langle 9\xi^2 - 6\xi c_1 + c_1^2, [M] \rangle - \langle 6\xi^2 - 4\xi + c_1^2 - c_2, [M] \rangle \\ &= \langle 3\xi^2 - 2\xi \cdot c_1 + c_2, [M] \rangle. \quad \text{q.e.d.} \end{aligned}$$

Using these and the Cayley-Zeuthen relations, we can obtain the geometrical invariants of a generic surface in  $CP_3$ . (See [4].) Let  $f: M \rightarrow CP_3$  be a generic surface, i.e., a cusp-free variety with  $\Gamma$  as double curve on which there are  $t$  triple-points which are also triple points of  $\Gamma$ , and with  $\nu$  points on  $\Gamma$  at which the two tangent planes coincide.

Let  $\epsilon_0$  be the order of  $\Gamma$ , i.e., the number of intersections with a generic plane, and  $\epsilon_1$  the class of  $\epsilon$ , i.e., the number of tangents with meet a generic line. Then

$$\begin{aligned} \epsilon_0 &= \{ \langle \xi^2, [M] \rangle \}^2 - \langle 2\xi^2 - \xi \cdot c_1, [M] \rangle, \\ \epsilon_1 &= \{ \langle \xi^2, [M] \rangle \}^2 - \langle \xi^2, [M] \rangle \cdot \langle \xi \cdot c_1, [M] \rangle \\ &\quad - \left\langle 21\xi^2 - 13\xi \cdot c_1 + \frac{7}{2}c_1^2 - \frac{5}{2}c_2, [M] \right\rangle, \\ t &= \binom{\langle \xi^2, [M] \rangle}{3} - \frac{1}{2} \langle \xi^2, [M] \rangle \langle \xi \cdot c_1, [M] \rangle \\ &\quad + \left\langle 8\xi^2 - \frac{14}{3}\xi \cdot c_1 + \frac{4}{3}c_1^2 - \frac{2}{3}c_2, [M] \right\rangle, \end{aligned}$$

and the number  $d$  of bitangents passing through a fixed point is

$$d = 12 \left( \frac{\langle \xi^2, [M] \rangle}{4} \right) - 4\varepsilon_0 \left( \frac{\langle \xi^2, [M] \rangle}{2} - 2 \right) - 2\varepsilon_1 - 12t + 2\varepsilon_0(\varepsilon_0 - 1).$$

**Remark.** It is instructive to note how radically different (and more complicated) these formulas are compared to the case of immersions in Euclidean space. For example, the number of triple points of an immersion of a compact 4-manifold in  $E^8$  is simply the topological index of the manifold—a topological invariant. This certainly is not so in projective geometry.

### Bibliography

- [1] F. Hirzebruch, *Topological methods in algebraic geometry*, Springer, Berlin, 1966.
- [2] W. F. Pohl, *Differential geometry of higher order*, *Topology* **1** (1962) 169–211.
- [3] ———, *Extrinsic complex projective geometry*, Proc. Conf. Complex Analysis (Minneapolis), Springer, Berlin 1965, 18–29.
- [4] J. G. Semple & S. L. Roth, *Introduction to algebraic geometry*, Oxford University Press, Oxford, 1949.

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